RATIONALITY OF THE PROBABILISTIC ZETA FUNCTION OF FINITELY GENERATED PROFINITE GROUPS

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ABSTRACT. We discuss whether finiteness properties of a profinite group G can be deduced from the probabilistic zeta function $P_G(s)$. In particular we prove that if $P_G(s)$ is rational and all but finitely many nonabelian composition factors of G are groups of Lie type in a fixed characteristic, then G contains only finitely many maximal subgroups.

1. INTRODUCTION

Let G be a finitely generated profinite group. As G has only finitely many open subgroups of a given index, for any $n \in \mathbb{N}$ we may define the integer $a_n(G)$ as $a_n(G) = \sum_H \mu_G(H)$, where the sum is over all open subgroups H of G with $|G : H| = n$. Here $\mu_G(H)$ denotes the Möbius function of the poset of open subgroups of G, which is defined by recursion as follows: $\mu_G(G) = 1$ and $\mu_G(H) = -\sum_{H<K} \mu_G(K)$ if $H < G$. Then we associate to G a formal Dirichlet series $P_G(s)$, defined as

$$P_G(s) = \sum_{n\in\mathbb{N}} \frac{a_n(G)}{n^s}.$$

Hall in [10] showed that if G is a finite group and t is a positive integer, then $P_G(t)$ is equal to the probability that t random elements of G generate G or in other words

$$P_G(t) = \mathrm{Prob}_G(t) := \frac{|\Omega_G(t)|}{|G^t|},$$

where $\Omega_G(t)$ is the set of generating t-tuples in G. In [13] Mann conjectured that $P_G(s)$ have a similar probabilistic meaning for a wide class of profinite groups. More precisely

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define $\mathrm{Prob}_G(t) = \mu(\Omega_G(t))$, where μ is the normalised Haar measure uniquely defined on the profinite group G^t and $\Omega_G(t)$ is the set of generating t-tuples in G (in the topological sense) and say that G is positively finitely generated if there exists a positive integer t such that $\mathrm{Prob}_G(t) > 0$. Mann conjectured that if G is positively finitely generated, then $P_G(s)$ converges in some right half-plane and $P_G(t) = \mathrm{Prob}_G(t)$, when $t \in \mathbb{N}$ is large enough. The second author proved in [12] that this conjecture is true if G a profinite group with polynomial subgroup growth. But even when the convergence is not ensured, the formal Dirichlet series $P_G(s)$ encodes information about the lattice generated by the maximal subgroups of G and combinatorial properties of the probabilistic sequence $\{a_n(G)\}$ reflect on the structure of G. For example in [8] it is proved that a finitely generated profinite group G is prosolvable if and only if the sequence $\{a_n(G)\}$ is multiplicative. Notice that if H is an open subgroup of G and $\mu_G(H) \neq 0$, then H is an intersection of maximal subgroups of G. This implies in particular that if G contains only finitely many maximal subgroups (i.e. if the Frattini subgroup $\mathrm{Frat}\, G$ of G has finite index in G), then there are only finitely many open subgroups H of G with $\mu_G(H) \neq 0$ and consequently $a_n(G) = 0$ for all but finitely many $n \in \mathbb{N}$ (i.e. $P_G(s)$ is a finite Dirichlet series). A natural question is whether the converse is true. An affirmative answer has been given in the case of prosolvable groups [6]. Really a stronger result holds: if G is a finitely generated prosolvable groups, then $P_G(s)$ is rational (i.e. $P_G(s) = A(s)/B(s)$ with $A(s)$ and $B(s)$ finite Dirichlet series) if and only if $G/\,\mathrm{Frat}\, G$ is a finite group. This has been generalized in [7] to the finitely generated profinite groups with the property that all but finite many factors in a composition series are either abelian or alternating groups. In this paper we prove two other results of the same nature.

Theorem 1. *Let G be a finitely generated profinite group. Assume that there exist a prime p and a normal open subgroup N of G such that the nonabelian composition factors of N are simple groups of Lie type over fields of characteristic p. Then $P_G(s)$ is rational if and only if $G/\mathrm{Frat}(G)$ is a finite group.*

Theorem 2. *Let G be a finitely generated profinite group. Assume that there exists a normal open subgroup N of G such that the nonabelian composition factors of N are sporadic simple groups. Then $P_G(s)$ is rational if and only if $G/\mathrm{Frat}(G)$ is a finite group.*

Together with the main result in [7], the two previous theorems implies:

Corollary. *Let G be a finitely generated profinite group. Assume that there exists a normal open subgroup N of G such that the nonabelian composition factors of N are all isomorphic. Then $P_G(s)$ is rational if and only if $G/\mathrm{Frat}(G)$ is a finite group.*

The idea of the proof is the following. In [4, 5] it is proved that $P_G(s)$ can be written as formal product $P_G(s) = \prod_i P_i(s)$ of finite Dirichlet series associated with the non-Frattini factors in a chief series of G. On the other hand $G/\,\mathrm{Frat}(G)$ is finite if and only if a chief series of G contains only finitely many non-Frattini factors. So the strategy is to prove that the product $\prod_i P_i(s)$ cannot be rational if it involves infinitely many non trivial factors. A consequence of the Skolem-Mahler-Lech Theorem (see Proposition 2.2) can help us in this task. However Proposition 2.2 concerns infinite product of finite Dirichlet series involving only one non trivial summand, but only the finite Dirichlet series associated to the abelian chief factors of G have this property, while in general the polynomials $P_i(s)$ are quite complicated. So we need to produce suitable "short" approximations $P_i^*(s)$ of the series $P_i(s)$, in such a way that the rationality of their product is preserved (see Proposition 2.3). This requires a delicate analysis of the subgroup structure of the almost simple groups

of Lie type, based in particular on the properties of the parabolic subgroups, and some information on the maximal subgroups of the sporadic simple groups.

2. INFINITE PRODUCTS OF FORMAL DIRICHLET SERIES

Let $\mathcal{R}$ be the ring of formal Dirichlet series with integer coefficients. We say that $F(s) = \sum_{n\in\mathbb{N}} a_n/n^s \in \mathcal{R}$ is a Dirichlet polynomial if $a_n = 0$ for all but finitely many $n \in \mathbb{N}$. The set $\mathcal{R}^*$ of the Dirichlet polynomials is a subring of $\mathcal{R}$. We will say that $F(s) \in \mathcal{R}$ is rational if there exist $A(s), B(s) \in \mathcal{R}^*$ with $F(s) = A(s)/B(s)$.

For every set π of prime numbers, we consider the ring endomorphism of $\mathcal{R}$ defined by:

$$F(s) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s} \mapsto F^\pi(s) = \sum_{n\in\mathbb{N}} \frac{a_n^*}{n^s}$$

where $a_n^* = 0$ if n is divisible by some prime $p \in \pi$, $a_n^* = a_n$ otherwise. We will use the following remark:

Remark 2.1. *For every set π of prime numbers, if $F(s)$ is rational then $F^\pi(s)$ is rational.*

The following result is a consequence of the Skolem-Mahler-Lech Theorem (see [6] for more details):

Proposition 2.2. *Let $I \subseteq \mathbb{N}$ and let q, r_i, c_i be positive integers for each $i \in I$. Assume that*

(i) *for every $n \in \mathbb{N}$, the set $\{i \in I \mid r_i \text{ divides } n\}$ is finite;*
(ii) *there exists a prime t such that t does not divide r_i for any $i \in I$.*

If the product

$$F(s) = \prod_{i\in I}\left(1 - \frac{c_i}{(q^{r_i})^s}\right)$$

is rational, then I is finite.

The following slight modification of Proposition 4.3 in [7] can be proved exactly in the same way and will play a relevant role in our arguments.

Proposition 2.3. *Let $F(s)$ be a product of finite Dirichlet series:*

$$F(s) = \prod_{i\in I} F_i(s), \text{ where } F_i(s) = \sum_{n\in\mathbb{N}} \frac{b_{i,n}}{n^s}$$

Let q be a prime and Λ the set of positive integers divisible by q. Assume that there exists a positive integer α and a set $\{r_i\}_{i\in I}$ of positive integers such that if $n \in \Lambda$ and $b_{i,n} \neq 0$ then n is an r_i-th power of some integer and $v_q(n) = \alpha r_i$ (where $v_q(n)$ is the q-adic valuation of n). Define

$$w = \min\{x \in \mathbb{N} \mid v_q(x) = \alpha \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in I\}.$$

If $F(s)$ is rational, then the product

$$F^*(s) = \prod_{i\in I}\left(1 + \frac{b_{i,w^{r_i}}}{(w^{r_i})^s}\right)$$

is also rational.

3. Preliminaries and notations

Let G be a finitely generated profinite group and let $\{G_i\}_{i\in\mathbb{N}}$ be a fixed countable descending series of open normal subgroups with the property that $G_0 = G$, $\cap_{i\in\mathbb{N}} G_i = 1$ and G_i/G_{i+1} is a chief factor of G/G_{i+1} for each $i \in \mathbb{N}$. In particular, for each $i \in \mathbb{N}$, there exist a simple group S_i and a positive integer r_i such that $G_i/G_{i+1} \cong S_i^{r_i}$. Moreover, as described in [5], for each $i \in \mathbb{N}$, a finite Dirichlet series

$$P_i(s) = \sum_{n\in\mathbb{N}} \frac{b_{i,n}}{n^s} \tag{3.1}$$

is associated with the chief factor G_i/G_{i+1} and $P_G(s)$ can be written as an infinite formal product of the finite Dirichlet series $P_i(s)$:

$$P_G(s) = \prod_{i\in\mathbb{N}} P_i(s). \tag{3.2}$$

Moreover, this factorization is independent on the choice of chief series (see [4, 5]) and $P_i(s) = 1$ unless G_i/G_{i+1} is a non-Frattini chief factor of G.

We recall some properties of the series $P_i(s)$. If S_i is cyclic of order p_i, then $P_i(s) = 1 - c_i/(p_i^{r_i})^s$, where c_i is the number of complements of G_i/G_{i+1} in G/G_{i+1}. It is more difficult to compute the series $P_i(s)$ when S_i is a non-abelian simple group. In that case a relevant role is played by the group $L_i = G/C_G(G_i/G_{i+1})$. This is a monolithic primitive group and its unique minimal normal subgroup is isomorphic to $G_i/G_{i+1} \cong S_i^{r_i}$. If $n \neq |S_i|^{r_i}$, then the coefficient $b_{i,n}$ in (3.1) depends only on the knowledge of L_i; more precisely we have

$$b_{i,n} = \sum_{\substack{|L_i:H|=n \\ L_i = H\mathrm{soc}(L_i)}} \mu_{L_i}(H).$$

It is not easy to compute these coefficients $b_{i,n}$ even for $n \neq |S_i|^{r_i}$. Some help comes from the knowledge of the subgroup X_i of $\mathrm{Aut}S_i$ induced by the conjugation action of the normalizer in L_i of a composition factor of the socle S_i^r (note that X_i is an almost simple group with socle isomorphic to S_i). More precisely, given an almost simple group X with socle S, we can consider the following Dirichlet polynomial:

$$P_{X,S}(s) = \sum_n \frac{c_n(X)}{n^s}, \text{ where } c_n(X) = \sum_{\substack{|X:H|=n \\ X=SH}} \mu_X(H). \tag{3.3}$$

The following can be deduced from [14]:

Lemma 3.1. *If S_i is nonabelian and π is a set of primes containing at least one divisor of $|S_i|$ then*

$$P_i^\pi(s) = P^\pi_{X_i,S_i}(r_i s - r_i + 1).$$

In particular, if n is not divisible by some prime in π, then there exists $m \in \mathbb{N}$ with $n = m^{r_i}$ and $b_{i,n} = c_m(X_i)\cdot m^{r_i-1}$.

We will give now a description of the polynomial $P^{\{p\}}_{X,S}(s)$ when S is a simple group of Lie type over a field of characteristic p and X is an almost simple group with socle S. We follow the notations from [1]. Recall that a simple group of Lie type S is the subgroup A^F of fixed points under a Frobenius map F of a connected reductive algebraic group A defined over an algebraically closed field of characteristic $p > 0$. In particular, S is defined over a field $\mathbb{K} = \mathbb{F}_q$ of characteristic p. To the map F, a symmetry ρ on the Dynkin

diagram of A^F is associated (ρ is trivial in the untwisted case). Let $I := \{\mathcal{O}_1, \cdots, \mathcal{O}_k\}$ be the set of the ρ-orbits on the nodes of the Dynkin diagram. For every subset $J \subseteq I$, let $J^* := \cup_{i\in J}\mathcal{O}_i$ be a ρ-stable subset of the set of nodes of the Dynkin diagram and one may associate an F-stable parabolic subgroup P_J of S with J^*. As described in [1, Chapter 9], we may associate to J a polynomial $T_{W_J}(x)$ with the property that $T_{W_J}(q) = |P_J|$. We have that:

Theorem 3.2. *Let S be a simple group of Lie type defined over a field $\mathbb{K} = \mathbb{F}_q$ of characteristic p and X an almost simple group with socle S. Then*

$$P_{X,S}^{\{p\}}(s) = (-1)^{|I|} \sum_{J\subseteq I} (-1)^{|J|} \left(\frac{T_W(s)}{T_{W_J}(s)}\right)^{1-s}.$$

In particular, if X does not contain non-trivial graph automorphisms, then

$$P_{X,S}^{\{p\}}(s) = P_S^{\{p\}}(s)$$

For later use we need to recall definitions and results concerning Zsigmondy primes.

Definition 3.3. *A prime number p is called a* primitive prime divisor *of $a^n - 1$ if it divides $a^n - 1$ but it does not divide $a^e - 1$ for any integer $1 \leq e \leq n-1$.*

The following theorem is due to K. Zsigmondy [15]:

Theorem 3.4 (Zsigmondy's Theorem)**.** *Let a and n be integers greater than 1. There exists a primitive prime divisor of $a^n - 1$ except exactly in the following cases:*

(1) $n = 2$, $a = 2^s - 1$, where $s \geq 2$.
(2) $n = 6$, $a = 2$.

Observe that there may be more than one primitive prime divisor of $a^n - 1$; we denote by $\langle a, n\rangle$ the set of these primes.

Let p be a prime, r a prime distinct from p and m an integer which is not a power of p. We define:

$$\zeta_p(r) = \min\{z \in \mathbb{N} \mid z \geq 1 \text{ and } p^z \equiv 1 \bmod r\},$$
$$\zeta_p(m) = \max\{\zeta_p(r) \mid r \text{ prime }, r \neq p, r|m\}.$$

The value of $\zeta_p(S) := \zeta_p(|S|)$ when S is a simple group of Lie type over $\mathbb{F}_q$ and $q = p^f$ is given in in [11, Table 5.2.C]).

Proposition 3.5. *Let X be an almost simple group with socle S, where S is a simple group of Lie type defined over a field of characteristic p. Assume that $\zeta_p(G) > 1$ and $\zeta_p(G) > 6$ if $p = 2$. Let $\tau \in \langle p, \zeta_p(S)\rangle$. Consider the Dirichlet series*

$$P_{X,S}^{\{p\}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

(a) *If $a_n \neq 0$ then τ divides n. More precisely, $v_\tau(n) = v_\tau(p^{\zeta_p(S)} - 1)$.*
(b) *If $m > \zeta_p(S)$ and a primitive prime divisor of $p^m - 1$ divides n, then $a_n = 0$.*
(c) *If n is the smallest positive integer such that $n \neq 1$ and $a_n \neq 0$, then $a_n < 0$.*

Proof. Statements (a) and (b) follows from Theorem 3.2 and the description of the polynomials $T_W(t)$ and $T_{W_J}(t)$ given in [1, Section 9.4, Section 14.2]. The crucial point is that if $q = p^f$, then f divides $\zeta_p(S)$ and $T_W(t)/T_{W_J}(t)$ is always divisible by the cyclotomic polynomial $\Phi_{\zeta_p(S)/f}(t)$ but non by $\Phi_m(t)$ for all $m > \zeta_p(S)/f$. Moreover, the smallest positive integer n such that $n \neq 0$ and $a_n \neq 0$ corresponds to J with $|I| - |J| = 1$. This implies immediately that $a_n < 0$, which proves (c). □

Combining the previous proposition with Lemma 3.1, we obtain:

Corollary 3.6. *Assume that $G_i/G_{i+1} \cong S_i^{r_i}$ is a chief factors of G, where S_i is a simple group of Lie type defined over a field of characteristic p. Assume that $\zeta_p(G) > 1$ and $\zeta_p(G) > 6$ if $p = 2$. If $\tau \in \langle p, \zeta_p(S_i)\rangle$, then we have:*

(a) *If $b_{i,n} \neq 0$ then τ divides n. More precisely, $v_\tau(n) = r_i \cdot v_\tau(p^{\zeta_p(S_i)} - 1)$.*
(b) *If $m > \zeta_p(S)$ and a primitive prime divisor of $p^m - 1$ divides n, then $b_{i,n} = 0$.*
(c) *If n is the smallest positive integer > 1 such that $(n, p) = 1$ and $b_{i,n} \neq 0$, then $b_{i,n} < 0$.*

4. Proofs of Theorem 1 and Theorem 2

We start now the proofs of our main results. We assume that G is a finitely generated profinite group G with the property that $P_G(s) = \sum_n a_n/n^s$ is rational. As described in Section 3, $P_G(s)$ can be written as a formal infinite product of Dirichlet polynomials $P_i(s) = \sum_{n\in\mathbb{N}} b_{i,n}/n^s$ corresponding to the factors G_i/G_{i+1} of a chief series of G. Let J be the set of indices i such that G_i/G_{i+1} is a non-Frattini chief factor. Since $P_i(s) = 1$ if $i \notin J$, we have

$$P_G(s) = \prod_{j\in J} P_j(s).$$

For $C(s) = \sum_{n=1}^{\infty} c_n/n^s \in \mathcal{R}$, we define $\pi(C(s))$ to be the set of the primes q for which there exists at least one multiple n of q with $c_n \neq 0$. Notice that if $C(s) = A(s)/B(s)$ is rational then $\pi(C(s)) \subseteq \pi(A(s)) \bigcup \pi(B(s))$ is finite. Let $\mathcal{S}$ be the set of the finite simple groups that are isomorphic to a composition factor of some non-Frattini chief factor of G. The first step in the proofs of Theorem 1 and 2 is to show that $\mathcal{S}$ is finite. The proof of this claim requires the following result.

Lemma 4.1 ([7, Lemma 3.1])**.** *Let G be a finitely generated profinite group and let q be a prime with $q \notin \pi(P_G(s))$. If q divides the order of a non-Frattini chief factor of G, then this factor is not a q-group.*

Let $\pi(G)$ be the set of the primes q with the properties that G contains at least an open subgroup H whose index is divisible by q.

Lemma 4.2. *If G satisfies the hypothesis of Theorem 1 or Theorem 2, then then the sets $\mathcal{S}$ and $\pi(G)$ are finite.*

Proof. Since $P_G(s)$ is rational, we have that $\pi(P_G(s))$ is finite. Therefore, it follows from Lemma 4.1 that $\mathcal{S}$ contains only finitely many abelian groups. If G satisfies the hypotheses of Theorem 2, then a non abelian group in $\mathcal{S}$ is either one of the 26 sporadic simple groups or is isomorphic to a composition factor of the finite group G/N. In any case we have only finitely many possibilities. Consider now the case when G satisfies the hypothesis of Theorem 1 and assume by contradiction that $\mathcal{S}$ is infinite. This is possible only if the subset $\mathcal{S}^*$ of the simple groups in $\mathcal{S}$ that are of Lie type over a field of characteristic p is infinite. In particular, the set $\Omega = \{\zeta_p(S) \mid S \in \mathcal{S}^*\}$ is infinite. Let

$$I := \{j \in J \mid S_j \in \mathcal{S}^*\},\ A(s) := \prod_{i\in I} P_i(s) \ \text{ and } \ B(s) := \prod_{i\notin I} P_i(s).$$

Notice that $\pi(B(s)) \subseteq \bigcup_{S\in\mathcal{S}\setminus\mathcal{S}^*} \pi(S)$ is a finite set. Since $P_G(s) = A(s)B(s)$ and $\pi(P_G(s))$ is finite, if follows that the set $\pi(A(s))$ is finite. In particular, there exists a positive integer $m \in \Omega$ such that $m \geq 7$ and $\langle p, m\rangle \cap \pi(A(s)) = \emptyset$ (here we choose

$m \geq 7$ to ensure, according with Theorem 3.4, that the set $\langle p, m\rangle$ is non empty). Let Γ_m be the set of the positive integers n such that no prime in $\langle p, u\rangle$ divides n if $u > m$ and set

$$\begin{aligned} r &:= \min\{r_i \mid S_i \in \mathcal{S}^* \text{ and } \zeta_p(S_i) = m\}, \\ I^* &:= \{i \in I \mid r_i = r \text{ and } S_i \in \mathcal{S}\}, \\ \beta &:= \min\{n > 1 \mid n \in \Gamma_m \text{ and } b_{i,n} \neq 0 \text{ for some } i \in I^*\}. \end{aligned}$$

By Corollary 3.6, if $i \in I$ and $b_{i,\beta} \neq 0$, then $\zeta_p(S_i) = m$, $r_i = r$ and $b_{i,\beta} < 0$. Hence the coefficient c_β of $1/\beta^s$ in $A(s)$ is

$$c_\beta = \sum_{i \in I, r = r_i} b_{i,\beta} = \sum_{i \in I^*} b_{i,\beta} < 0.$$

On the other hand, again by Corollary 3.6, all the primes in $\langle p, m\rangle$ divides m. But then $\langle p, m\rangle \subseteq \pi(A(s))$, which is a contradiction. So we have proved that $\mathcal{S}$ is finite. By [7, Lemma 3.2], if follows that $\pi(G)$ is also finite. □

The previous result allows us to employ the following:

Proposition 4.3. *[7, Corollary 5.2] Let G be a finitely generated profinite group and assume that $\pi(G)$ is finite. For each n, there are only finitely many non-Frattini factors in a chief series whose composition length is at most n. Moreover then there exists a prime t such that no non-Frattini chief factor of G has composition length divisible by t.*

Combined with Proposition 2.2, the previous result implies:

Corollary 4.4. *Let G be a finitely generated profinite group, assume that $\pi(G)$ is finite and let r_i be the sequence of the composition lengths of the non-Frattini factors in a chief series of G. Assume that there exists a positive integer q and a sequence c_i of nonnegative integers such that the formal product*

$$H(s) = \prod_i \left(1 - \frac{c_i}{(q^{r_i})^s}\right)$$

is rational. Then $c_i = 0$ for all but finitely many indices i.

For a simple group $S \in \mathcal{S}$, let $I_S = \{j \in J \mid S_j \cong S\}$. Our aim is to prove that, in the hypotheses of Theorem 1 and 2, J is a finite set. We have already proved that $\mathcal{S}$ is finite, so it suffices to prove that I_S is finite for each $S \in \mathcal{S}$. First we consider the case when S is abelian.

Lemma 4.5. *Assume that G satisfies the hypothesis of Theorem 1 or Theorem 2. If q is a prime and S is cyclic of order q, then I_S is a finite set.*

Proof. Let $\mathcal{S}_q$ be the set of the non abelian simple groups in $\mathcal{S}$ containing a proper subgroup of q-power index. A theorem proved by Guralnick [9] implies that if $T \in \mathcal{S}_q$ then there exists a unique positive integer $\alpha(T)$ with the property that T contains a subgroup of index $q^{\alpha(T)}$. Consider the set π of all the primes different from q. By [3, Lemma 14], there exist positive integers c_i and nonnegative integers d_i such that

$$P_G^\pi(s) = \prod_{i \in I_S} \left(1 - \frac{c_i}{q^{r_i s}}\right) \prod_{T \in \mathcal{S}_q} \left(\prod_{j \in I_T} \left(1 - \frac{d_j}{q^{\alpha(T) r_j s}}\right)\right) \tag{4.1}$$

Since $\mathcal{S}$ is finite, the set $\{\alpha(T) \mid T \in \mathcal{S}_q\}$ is finite. Moreover, by Proposition 4.3, there is a prime number t such that no element in

$$\{r_i \mid i \in I_S\} \bigcup \{\alpha(T) r_j \mid T \in \mathcal{S}_q \text{ and } j \in I_T\}$$

is divisible by t. Since $P_G(s)$ is rational, $P_G^{\pi}(s)$ is also rational. But then, by Proposition 2.2, the number of nontrivial factors in the product at the right side of equation (4.1) is finite. In particular, I_S is a finite set. $\square$

Proof of Theorem 1. Let $\mathcal{T}$ be the set of the almost simple groups X such that there exist infinitely many $i \in J$ with $X_i \cong X$ and let $I = \{i \in J \mid X_i \in \mathcal{T}\}$. The hypotheses of Theorem 1 combined with Lemma 4.5 implies that $J \setminus I$ is finite. We have to prove that J is finite; this is equivalent to show that $I = \emptyset$. But then, in order to complete our proof, it suffices to prove the following claim.
$(*)$ *For every* $n \in \mathbb{N}$, $I_n = \{i \in I \mid \zeta_p(S_i) = n\} = \emptyset$.
Assume that the claim is false and let m be the smallest integer such that the set $I_m \neq \emptyset$. Since $J \setminus I$ is finite and $P_G(s) = \prod_{i \in J} P_i(s)$ is rational, also $\prod_{i \in I} P_i(s)$ is rational. In particular, the following series is rational:

$$Q(s) = \prod_{i \in I} P_i^{\{p\}}(s).$$

We distinguish three different cases:

(1) $m = 1, p = 2^t - 1, t \geq 2$;
(2) $m \leq 5, p = 2$;
(3) all the other possibilities.

In cases (1) and (3), it follows by Theorem 3.4 that $\langle p, t\rangle \neq \emptyset$ for every $t > m+1$; we set $\pi = \bigcup_{t > m+1} \langle p, t\rangle \cup \{p\}$. In case (2), $\langle p, t\rangle \neq \emptyset$ whenever $t > 6$ and we set $\pi = \bigcup_{t>6} \langle p, t\rangle \cup \{p\}$. The Dirichlet series $H(s) = Q^{\pi}(s)$ is rational. By Corollary 3.6, if $i \in I_t$ and $\tau \in \langle p, t\rangle$, then $P_i^{\{\tau,p\}}(s) = 1$; in particular $P_i^{\pi}(s) = 1$ whenever $\langle p, t\rangle \subseteq \pi$. This implies

$$H(s) = \begin{cases} \prod_{i \in I_m} P_i^{\{p\}}(s) & \text{in cases (1) and (3) ,} \\ \prod_{\substack{i \in I_u \\ m \leq u \leq 5}} P_i^{\{2\}}(s) & \text{otherwise.} \end{cases}$$

Assume that case (3) occurs and let $\tau \in \langle p, m\rangle$. By Lemma 3.1 and Corollary 3.6, if $i \in I_m$, $(p, y) = 1$ and $b_{i,y} \neq 0$, then $y = x^{r_i}$ and $v_\tau(x) = v_\tau(p^m - 1)$. Let

$$w = \min\{x \in \mathbb{N} \mid v_\tau(x) = v_\tau(p^m - 1) \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in I_m\}.$$

By Corollary 3.6, for each $i \in I_m$, if $b_{i,w^{r_i}} \neq 0$ then $b_{i,w^{r_i}} < 0$. Moreover if $b_{i,w^{r_i}} \neq 0$ and $X_j \cong X_i$, then $b_{j,w^{r_j}} \neq 0$, so the set $\Sigma_m = \{i \in I_m \mid b_{i,w^{r_i}} \neq 0\}$ is infinite. Applying Proposition 2.3, we obtain a rational product

$$H^*(s) = \prod_{i \in \Sigma_m} \left(1 + \frac{b_{i,w^{r_i}}}{w^{r_i s}}\right), \text{ where } b_{i,w^{r_i s}} < 0 \text{ for all } i \in \Sigma_m.$$

By Corollary 4.4, $H^*(s)$ is a finite product, i.e. Σ_m is finite, which is a contradiction.

Assume now that (1) occurs. By [11, Table 5.2.C] if $\zeta_p(S) = 1$, then $S \cong PSL_2(p)$. This implies in particular that

$$H(s) = \prod_{i \in I_1} \left(1 - \frac{2^{tr_i}}{2^{tr_i s}}\right).$$

By Corollary 4.4, we get that I_1 is finite, which is a contradiction.

Finally assume that case (2) occurs. If $\zeta_p(S) \leq 5$, then S is one of the following groups: $PSL_6(2), U_4(2), PSp_6(2), P\Omega_8^+(2), PSL_3(4), SL_5(2), PSL_4(2), PSL_3(2)$. The explicit description of the Dirichlet series $P_{X,S}^{\{2\}}(s)$ when $S \leq X \leq \mathrm{Aut}(S)$ and S is one of the simple groups in the previous list is included in the Appendix 1. Notice in particular that if $i \in \Lambda = \bigcup_{m \leq 5} I_m$ then $\pi(P_i^{\{2\}}(s)) \subseteq \{3, 7, 5, 31\}$. First consider

$$\Lambda_{31} = \{i \in \Lambda \mid 31 \in \pi(P_i^{\{2\}}(s))\}$$

and let

$$\begin{aligned} w &= \min\{x \in \mathbb{N} \mid x \text{ is odd}, v_{31}(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in \Lambda\} \\ &= \min\{x \in \mathbb{N} \mid x \text{ is odd}, v_{31}(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in \Lambda_{31}\}. \end{aligned}$$

Note that if $i \in \Lambda_{31}$ and n is minimal with the properties that n is odd, $b_{i,n^{r_i}} \neq 0$ and $v_{31}(n) = 1$, then $b_{i,n^{r_i}} < 0$ (see Appendix 1). So if $b_{i,w^{r_i}} \neq 0$ then $b_{i,w^{r_i}} < 0$; moreover, by applying Proposition 2.3, we obtain a rational product

$$H^*(s) = \prod_{i \in \Lambda} \left(1 + \frac{b_{i,w^{r_i}}}{w^{r_i s}}\right) = \prod_{i \in \Lambda_{31}} \left(1 + \frac{b_{i,w^{r_i}}}{w^{r_i s}}\right), \text{ where } b_{i,w^{r_i}} \leq 0 \text{ for all } i \in \Lambda_{31}.$$

By Corollary 4.4, the set $\Lambda_{31}^* = \{i \in \Lambda_{31} \mid b_{i,w^{r_i}} \neq 0\}$ is finite, but this implies that $\Lambda_{31} = \emptyset$. Indeed if $\Lambda_{31} \neq \emptyset$ then there exists at least one index i with $i \in \Lambda_{31}^*$, moreover by assumption there are infinitely many j with $X_j \cong X_i$ and all of them belong to Λ_{31}^*. Since $\Lambda_{31} = \emptyset$, if $i \in \Lambda$, then S_i is isomorphic to one of the following: $U_4(2), PSp_6(2), P\Omega_8^+(2), PSL_3(4), PSL_4(2), PSL_3(2)$. It follows from Appendix 1, that if $i \in \Lambda$, x is odd and $b_{i,x^{r_i}} \neq 0$, then $v_7(x) \leq 1$. But then, we may repeat the same argument as above and consider $\Lambda_7 = \{i \in \Lambda \mid 7 \in \pi(P_i^{\{2\}}(s))\}$ and

$$w := \min\{x \in \mathbb{N} \mid x \text{ is odd}, v_7(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in \Lambda_7\}.$$

Arguing as before we deduce that $\Lambda_7 = \emptyset$. We can see from Appendix 1 that this implies $S_i \cong U_4(2)$ for all $i \in \Lambda$ and

$$H^{\{5\}}(s) = \prod_{i \in \Lambda} \left(1 - \frac{3^{3r_i}}{3^{3r_i s}}\right).$$

Again, by Corollary 4.4, Λ is finite and consequently $\Lambda = \emptyset$. □

Proof of Theorem 2. Let $\mathcal{T}$ be the set of the almost simple groups X such that $\mathrm{soc}X$ is a sporadic simple groups and there exist infinitely many $i \in J$ with $X_i \cong X$ and let $I = \{i \in J \mid X_i \in \mathcal{T}\}$. As in the case of Theorem 1, we have to prove that $I = \emptyset$. For an almost simple group X, let $\Omega(X)$ be the set of the odd integers $m \in \mathbb{N}$ such that

- X contains at least one subgroup Y such that $X = Y\mathrm{soc}X$ and $|X : Y| = m$;
- if $X = Y\mathrm{soc}X$ and $|X : Y| = m$, then Y is a maximal subgroup if X.

Note that if $m \in \Omega(X)$, $X = Y\mathrm{soc}X$ and $|X : Y| = m$, then $\mu_X(Y) = -1$: in particular $c_m(X) < 0$. Combined with Lemma 3.1, this implies that if $m \in \Omega(X_i)$ then $b_{i,m^{r_i}} < 0$. Certainly $\Omega(X)$ is not empty and its smallest element is the smallest index $m(X)$ of a supplement of $\mathrm{soc}X$ in X containing a Sylow 2-subgroup of X. When $S = \mathrm{soc}X$ is a sporadic simple group, the value of $m(X)$ can be read from [2]; the precise values are given in Table 1. In few cases we need to know another integer $n(X)$ in $\Omega(X)$, given in Table 2. For a fixed prime p, let $\Lambda_p = \{i \in I \mid p \in \pi(P_i^{\{2\}}(s))\}$.

If $i \in \Lambda_{31}$, then 31 divide $|S_i|$ and $S_i \in \{\mathrm{J}_4, \mathrm{Ly}, \mathrm{O'N}, \mathrm{BM}, \mathrm{M}, \mathrm{Th}\}$. Moreover 31^2 does not divide $|S_i|$ so if n is odd, divisible by 31 and $b_{i,n} \neq 0$ then $n = x^{r_i}$ and $v_{31}(x) = 1$.

Let $m_i = n(S_i)$ if $S_i \cong$ Th, $m_i = m(S_i)$ otherwise. Since m_i is the smallest odd number divisible by 31 and equal to the index in X_i of a supplement of S_i we get:

$$\begin{aligned} w &= \min\{x \in \mathbb{N} \mid x \text{ is odd }, v_{31}(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in I\} \\ &= \min\{x \in \mathbb{N} \mid x \text{ is odd }, v_{31}(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in \Lambda_{31}\} \\ &= \min\{m_i \mid i \in \Lambda_{31}\}. \end{aligned}$$

But then by Proposition 2.3, the following Dirichlet series is rational:

$$\prod_{i \in \Lambda_{31}} \left(1 + \frac{b_{i,w^{r_i}}}{(w^{r_i})^s}\right).$$

We have $b_{i,w^{r_i}} < 0$ if $m_i = w$, $b_{i,w^{r_i}} = 0$ otherwise. By applying Corollary 4.4, we get that $\{i \in \Lambda_{31} \mid m_i = w\}$ is a finite set, and this implies $\Lambda_{31} = \emptyset$.

Now consider Λ_{23}. Since $\Lambda_{31} = \emptyset$, if $i \in \Lambda_{23}$ then $S_i \in \{\mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{Co}_1, \mathrm{Co}_2, \mathrm{Co}_3, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}'\}$. We can repeat the argument used to proved that $\Lambda_{31} = \emptyset$. Let $m_i = n(S_i)$ if $S_i \cong \mathrm{Co}_1$, $m_i = m(X_i)$ otherwise and let $w = \min\{m_i \mid i \in \Lambda_{23}\}$. By applying Corollary 4.4, we get that $\{i \in \Lambda_{23} \mid m_i = w\}$ is a finite set, and this implies $\Lambda_{23} = \emptyset$.

New we consider Λ_{11}. Since $\Lambda_{31} \cup \Lambda_{23} = \emptyset$, if $i \in \Lambda_{11}$, then $S_i \in \{\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{J}_1, \mathrm{HS}, \mathrm{Suz}, \mathrm{McL}, \mathrm{HN}, \mathrm{Fi}_{22}\}$. Let $m_i = n(X_i)$ if $S_i \cong \mathrm{Fi}_{22}$ or $S_i \cong \mathrm{Fi}_{24}'$, $m_i = m(X_i)$ otherwise and let $w = \min\{m_i \mid i \in \Lambda_{11}\}$. As before, by applying Corollary 4.4, we get that $\{i \in \Lambda_{23} \mid m_i = w\}$ is a finite set, and this implies $\Lambda_{11} = \emptyset$. Continuing our procedure, we consider Λ_{17}: if $i \in \Lambda_{17}$, then $S_i \in \{\mathrm{J}_3, \mathrm{He}\}$ and we can take $w = \min\{m(X_i) \mid i \in \Lambda_{17}\}$ and deduce that $\Lambda_{17} = \emptyset$. Next we take $w = m(\mathrm{Ru})$ to prove $\Lambda_{29} = \emptyset$ and finally we take $w = m(\mathrm{J}_2)$ to prove $\Lambda_7 = \emptyset$. □

Appendix: exceptional cases

In this section, we give explicit formula for $P^{(2)}_{X,S}(s)$ when X is an almost simple group whose socle S is of Lie type over a field of characteristic 2 and $\zeta_2(S) \leq 6$.

(i) $S = \mathrm{PSL}_6(2)$. If X contains a graph automorphism then

$$\begin{aligned} P^{(2)}_{X,S}(s) &= 1 - (3^2 \cdot 7 \cdot 31)^{(1-s)} - (3 \cdot 5 \cdot 7^2 \cdot 31)^{(1-s)} - (3^3 \cdot 7 \cdot 31)^{(1-s)} + \\ &\quad + 2(3^4 \cdot 7^2 \cdot 31)^{(1-s)} + (3^3 \cdot 5 \cdot 7^2 \cdot 31)^{(1-s)} - (3^4 \cdot 5 \cdot 7^2 \cdot 31)^{(1-s)}. \end{aligned}$$

If X does not contain graph automorphisms, then

$$\begin{aligned} P^{(2)}_{X,S}(s) &= 1 - 2(3^2 \cdot 7)^{(1-s)} - (3^2 \cdot 5 \cdot 31)^{(1-s)} - 2(3 \cdot 7 \cdot 31)^{(1-s)} + \\ &\quad + 3(3^2 \cdot 7 \cdot 31)^{(1-s)} + 6(3^2 \cdot 5 \cdot 7 \cdot 31)^{(1-s)} + (3 \cdot 5 \cdot 7^2 \cdot 31)^{(1-s)} - \\ &\quad - 4(3^3 \cdot 5 \cdot 7 \cdot 31)^{(1-s)} - 6(3^2 \cdot 5 \cdot 7^2 \cdot 31)^{(1-s)} + \\ &\quad + 5(3^3 \cdot 5 \cdot 7^2 \cdot 31)^{(1-s)} - (3^4 \cdot 5 \cdot 7^2 \cdot 31)^{(1-s)}. \end{aligned}$$

(ii) $S = \mathrm{PSL}_5(2)$. If X contains a graph automorphism then

$$P^{(2)}_{X,S}(s) = 1 - (3 \cdot 5 \cdot 31)^{(1-s)} - (3^2 \cdot 7 \cdot 31)^{(1-s)} + (3^2 \cdot 5 \cdot 7 \cdot 31)^{(1-s)}.$$

If X does not contain graph automorphisms, then

$$\begin{aligned} P^{(2)}_{X,S}(s) &= 1 - 2(31)^{(1-s)} - 2(5 \cdot 31)^{(1-s)} + 3(3 \cdot 5 \cdot 31)^{(1-s)} + \\ &\quad + 3(5 \cdot 7 \cdot 31)^{(1-s)} - 4(3 \cdot 5 \cdot 7 \cdot 31)^{(1-s)} + (3^2 \cdot 5 \cdot 7 \cdot 31)^{(1-s)}. \end{aligned}$$

(iii) $S = \mathrm{PSL}_4(2)$. If X contains a graph automorphism then

$$P^{(2)}_{X,S}(s) = 1-(3^2\cdot 7)^{(1-s)}-(3\cdot 5\cdot 7)^{(1-s)}+(3^2\cdot 5\cdot 7)^{(1-s)}.$$

If X does not contain graph automorphisms, then

$$P^{(2)}_{X,S}(s) = 1-2(3\cdot 5)^{(1-s)}-(5\cdot 7)^{(1-s)}+3(3\cdot 5\cdot 7)^{(1-s)}-(3^2\cdot 5\cdot 7)^{(1-s)}.$$

(iv) $S = \mathrm{PSL}_3(2)$. If X contains a graph automorphism then

$$P^{(2)}_{X,S}(s) = 1-(3\cdot 7)^{(1-s)}.$$

If X does not contain graph automorphisms, then

$$P^{(2)}_{X,S}(s) = 1-2(7)^{(1-s)}+(3\cdot 7)^{(1-s)}.$$

(v) $S = \mathrm{PSL}_3(4)$. If X contains a graph automorphism then

$$P^{(2)}_{X,S}(s) = 1-(3\cdot 5\cdot 7)^{(1-s)}.$$

If X does not contain graph automorphisms then

$$P^{(2)}_{X,S}(s) = 1-2(3\cdot 7)^{(1-s)}+(3\cdot 5\cdot 7)^{(1-s)}.$$

(vi) $S = \mathrm{PSp}_6(2)$. We have

$$P^{(2)}_{X,S}(s) = 1-(3^2\cdot 7)^{(1-s)}-(3^3\cdot 5)^{(1-s)}-(3^2\cdot 5\cdot 7)^{(1-s)}+3(3^3\cdot 5\cdot 7)^{(1-s)}-(3^4\cdot 5\cdot 7)^{(1-s)}.$$

(vii) $S = \mathrm{U}_4(2)$. We have

$$P^{(2)}_{X,S}(s) = 1-(3^3)^{(1-s)}-(3^2\cdot 5)^{(1-s)}+(3^3\cdot 5)^{(1-s)}.$$

(viii) $S = \mathrm{P}\Omega_8^+(2)$. We have

$$\begin{aligned} P^{(2)}_{X,S}(s) &= 1-3(3^2\cdot 5)^{(1-s)}-(3\cdot 5^2\cdot 7)^{(1-s)}+3(3^3\cdot 5^2)^{(1-s)}+ \\ &\quad +3(3^3\cdot 5^2\cdot 7)^{(1-s)}-4(3^4\cdot 5^2\cdot 7)^{(1-s)}+(3^5\cdot 5^2\cdot 7)^{(1-s)}. \end{aligned}$$

Table 1: Sporadic simple groups

X	$\lvert X\rvert$	$m(X)$
M_{11}	$2^4\cdot 3^2\cdot 5\cdot 11$	11
M_{12}	$2^6\cdot 3^3\cdot 5\cdot 11$	$3^2\cdot 5\cdot 11$
$\mathrm{Aut}(\mathrm{M}_{12})$	$2^7\cdot 3^3\cdot 5\cdot 11$	$3^2\cdot 5\cdot 11$
M_{22}	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$	$7\cdot 11$
$\mathrm{Aut}(M_{22})$	$2^8\cdot 3^2\cdot 5\cdot 7\cdot 11$	$7\cdot 11$
M_{23}	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$	23
M_{24}	$2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$	$3\cdot 11\cdot 23$
J_1	$2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19$	$5\cdot 11\cdot 19$
J_2	$2^7\cdot 3^3\cdot 5^2\cdot 7$	$3^2\cdot 5\cdot 7$
$\mathrm{Aut}(\mathrm{J}_2)$	$2^8\cdot 3^3\cdot 5^2\cdot 7$	$3^2\cdot 5\cdot 7$
J_3	$2^7\cdot 3^5\cdot 5\cdot 17\cdot 19$	$3^4\cdot 17\cdot 19$
$\mathrm{Aut}(\mathrm{J}_3)$	$2^8\cdot 3^5\cdot 5\cdot 17\cdot 19$	$3^4\cdot 17\cdot 19$
J_4	$2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3\cdot 23\cdot 29\cdot 31\cdot 37\cdot 43$	$11^2\cdot 29\cdot 31\cdot 37\cdot 43$
HS	$2^9\cdot 3^2\cdot 5^3\cdot 7\cdot 11$	$3\cdot 5^3\cdot 11$
Aut(HS)	$2^{10}\cdot 3^2\cdot 5^3\cdot 7\cdot 11$	$3\cdot 5^3\cdot 11$
Suz	$2^{13}\cdot 3^7\cdot 5^2\cdot 7\cdot 11\cdot 13$	$3^3\cdot 5\cdot 7\cdot 11\cdot 13$

$\mathrm{Aut}(\mathrm{Suz})$	$2^{14} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	$5^2 \cdot 11$
$\mathrm{Aut}(\mathrm{McL})$	$2^8 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	$5^2 \cdot 11$
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	$3^2 \cdot 5^3 \cdot 13 \cdot 29$
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	$5 \cdot 7^3 \cdot 17$
$\mathrm{Aut}(\mathrm{He})$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	$3^2 \cdot 5^2 \cdot 7^2 \cdot 17$
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	$5^3 \cdot 31 \cdot 37 \cdot 67$
O'N	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	$3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
$\mathrm{Aut}(\mathrm{O'N})$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	$3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	$3^6 \cdot 5^3 \cdot 7 \cdot 13$
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	$3^4 \cdot 5^2 \cdot 23$
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	$3^3 \cdot 5^2 \cdot 11 \cdot 23$
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$3^7 \cdot 5 \cdot 13$
$\mathrm{Aut}(\mathrm{Fi}_{22})$	$2^{18} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$3^7 \cdot 5 \cdot 13$
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	$3^4 \cdot 17 \cdot 23$
Fi_{24}'	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	$3^{13} \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 29$
$\mathrm{Aut}(\mathrm{Fi}_{24}')$	$2^{22} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 29$	$3^{13} \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 29$
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	$3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19$
$\mathrm{Aut}(\mathrm{HN})$	$2^{15} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	$3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19$
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19$
BM	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	$3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 47$
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29$ $\cdot 31 \cdot 41 \cdot 59 \cdot 71$	$3^{11} \cdot 5^5 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot$ $\cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

Table 2: $n(X)$

X	$n(X)$
Co_1	$3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23$
Fi_{22}	$3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
Fi_{24}'	$3^9 \cdot 5 \cdot 11 \cdot 7^2 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$\mathrm{Aut}(\mathrm{Fi}_{24}')$	$3^9 \cdot 5 \cdot 11 \cdot 7^2 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
Th	$3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 31$

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